Technical Comments

Comment on "Finite Element Structural Analysis of Local Instability"

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THE use of wavelength dependent finite elements in the form of strips, used by Przemieniecki[1] to determine local buckling stresses of thin sections in compression, is a restricted application of the now well-known finite strip method which has been extensively developed by Y. K. Cheung. Cheung has applied the method to a variety of structural problems and has included in-plane deformations of the component flats as well as the bending deformations considered by Przemieniecki. Thus, Cheung's strip elements are capable of being used to calculate buckling loads corresponding to longer wavelength modes in which the line junctions between adjacent flats do not remain straight, as in torsional or over-all buckling of stiffened panels. Recently one of the writer's research students, R. J. Plank, has used Cheung's type of strip element to solve problems of the buckling of thin-walled sections subjected to loads which give rise to combined longitudinal and transverse direct stress, in-plane shear, and longitudinal in-plane bending in the component flats. The inclusion of the shear loading implies that the buckling displacements of the various junctions will not all be in-phase and this results in a stiffness matrix which is complex Hermitian rather than real and symmetrical. A paper describing this work has been prepared.

This type of problem has been studied extensively by the writer and his colleagues for the past seven years using a somewhat different approach. If, as in Przemieniecki's paper, the assumption is made that all three components of displacement of any longitudinal line vary sinusoidally with the longitudinal coordinate, the differential equations governing in-plane and out-of-plane displacements of a component flat can be solved exactly by separation of variables and stiffness matrices derived for the component flats.[2,3] These relate the sinusoidally varying perturbation forces and displacements on their two longitudinal edges and take full account of the destabilising influence of the basic longitudinal compression, not only in the out-of-plane but also in the in-plane sense. The latter is imperative if meaningful results are to be obtained for the buckling stresses in longer wavelength modes which involve in-plane motions of some of the component flats, as in torsional or over-all modes of Z-stiffened panels for example.

References 2 and 3 give explicit expressions for the stiffness matrices when the component flats are isotropic and are subjected to a combination of uniform longitudinal compression and shear. As mentioned previously the inclusion of shear leads to complex Hermitian stiffness matrices. More recently, explicit expressions have been derived for the stiffness matrices of anisotropic plates, relevant to composite construction, when each plate carries uniform longitudinal compression (or tension), transverse compression (or tension), and shear, and a paper on this work is in preparation. This theory has been incorporated into a general purpose computer programme called VIPASA, which is capable of computing automatically the load factors at buckling, or the natural frequencies of vibration, of thin structures consisting of an assembly of flats of this general type.

Przemieniecki[1] pointed out that the elements of the stiffness matrices derived by this approach are transcendental functions of the stresses and the wavelength. This is quite correct, but he also went on to say that "the computational procedure for obtaining the buckling stress is very time consuming." The writer wishes to question this statement and the following comments are based upon the experience that has been gained at Birmingham University.

First, it is certainly true that the stiffness matrices of individual elements using the approximate finite strip approach involve simple algebraic expressions, whilst those derived by the exact approach involve much more complicated transcendental expressions. Thus, the time taken to set up the element stiffness matrices at a given value of the load factor is certainly less in the approximate than the exact approach. However, the exact approach scores over the approximate one because fewer elements are required. In order to achieve adequate accuracy with the approximate approach it is certainly necessary to subdivide each component flat into at least two, and in some cases more, strips, whereas in the exact approach each component flat can be treated as a single element. Thus, the order of the over-all stiffness matrix in the approximate approach is at least twice that in the exact approach. Moreover, because intermediate nodes have had to be introduced, it usually happens that the minimum possible bandwidth of the approximate over-all stiffness matrix is greater than that of the exact one. These two factors combined can more than outweigh the saving of time resulting from the algebraic rather than the transcendental element stiffness matrices, and indeed our experience is that for those problems in which we have direct comparison, the exact approach is in fact somewhat faster than the approximate one.

It may be argued, in its favor, that the approximate approach leads to an eigenvalue problem of standard type, enabling standard eigenvalue solving routines to be used. However, the fact that the exact approach leads to a nonstandard eigenvalue problem, with the load factor contained in the matrix in transcendental form, is not an obstacle in view of the existence of the algorithm derived by the writer and his colleague, Dr. F. W. Williams.[4,5] This is essentially an extension of the Sturm sequence procedure for the standard eigenvalue problem and provides an automatic way of converging on the eigenvalues that is no more time-consuming than the Sturm sequence procedure[6] for the standard problem. Moreover, it provides great flexibility in enabling one to assemble the complete structure from substructures without any of the loss of accuracy that is inherent in eigenvalue economizer techniques. This is a great advantage if the structure contains a number of identical substructures, a feature that is often present in stiffened panels and which cannot otherwise be used to effect economies in processing the over-all stiffness matrix.

There is one other minor comment, relating to Przemieniecki's paper, that can be made. Figure 5 shows the interaction curve between longitudinal and transverse stress for buckling of a long plate with simply supported edges under biaxial stress. It compares the results of the finite strip approximation to the exact analytical curve and shows some of the approximate results as lying slightly below the exact curve. The writer feels that this must be incorrect, since the finite strip method must give upper bounds.

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References

[1] Przemieniecki, J. S., "Finite Element Structural Analysis of Local Instability," *AIAA Journal*, Vol. 11, No. 1, Jan. 1973, pp. 33–39.

[2] Wittrick, W. H., "A Unified Approach to the Initial Buckling of Stiffened Panels in Compression," *The Aeronautical Quarterly*, Vol. 19, Part 3, Aug. 1968, pp. 265–283.

[3] Wittrick, W. H., "General Sinusoidal Stiffness Matrices for Buckling and Vibration Analysis of Thin Flat-Walled Structures," *International Journal of Mechanical Sciences*, Vol. 10, 1968, pp. 949–966.

[4] Wittrick, W. H. and Williams, F. W., "A General Algorithm for Computing Natural Frequencies of Elastic Structures," *Quarterly Journal of Mechanics and Applied Maths*, Vol. 14, 1971, pp. 263–284.

[5] Wittrick, W. H. and Williams, F. W., "An Algorithm for Computing Critical Buckling Loads of Elastic Structures," *Journal of Structural Mechanics*, Vol. 1, 1973, pp. 497–518.

[6] Peters, G. and Wilkinson, J. H., "Eigenvalues of $Ax = \lambda Bx$ with Band Symmetric A and B," *The Computer Journal*, Vol. 12, 1969, pp. 398–404.

Reply by Author to W. H. Wittrick

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THE discussion of current work by Wittrick and his associates at Birmingham University is of a great interest. Not many exact solutions to complicated buckling problems are available and this work will be helpful in comparing the finite element solutions with those obtained by solving the nonlinear equations for elastic thin-walled plate structures subjected to compressive loading.

Wittrick's comment that the finite element stiffness in Ref. 1 is a special case of his exact stiffness calls for a further clarification in order to put it in proper perspective. Coefficients in the approximate stiffness matrices could, in fact, be derived by expanding the exact coefficients represented by a quotient of two complicated transcendental functions of the compressive stress σ and the half-wavelength λ and retaining only the first order terms, a monumental task in exercising one's algebraic dexterity. Only through such a manipulation can any connection between Wittrick's stiffnesses and the finite element elastic and geometric stiffnesses be established. It should also be pointed out that the two methods are based on entirely different approaches. Wittrick uses an analytical solution of the nonlinear theory of elasticity while Ref. 1 uses the standard finite element approach based on the concept of geometrical stiffness. Thus, the author's method is simply an extension of the conventional finite element technique to a special class of problems involving local instability. There is still another fundamental difference between the two methods: Wittrick's method uses the stability determinant to obtain the buckling stress which contrasts with the author's use of the standard eigenvalue equations.

The author's comment that the computational procedure for obtaining the buckling stress from the exact solution is very time consuming was simply made on the basis of the comparison of typical elements in the stiffness matrices of the two methods. For example, the stiffness coefficient for the out-of-plane plate edge rotation in the exact method, using notation of Ref. 2, is given by

$$s_{MM} = \frac{D(\xi)^{1/2}}{b} \frac{\alpha \cosh \alpha \sinh \gamma - \gamma \cosh \gamma \sinh \alpha}{\sinh a \sinh \gamma + (\alpha\gamma/\omega^2)(1 - \cosh \alpha \cosh \gamma)} \quad (1)$$

for $\xi < 1$ while for $\xi > 1$ and $\xi = 1$ other similar transcendental expressions are used. In these expressions ξ, α, and γ are functions of the stress σ and the half-wavelength λ and ω depends on only λ. Equation (1) may be compared with the corresponding finite element stiffness coefficient expressed in algebraic form as

$$k_{22} = 2D\left[\frac{\pi^4}{420}\left(\frac{b}{\lambda}\right)^3 + \frac{\pi^2}{15}\left(\frac{b}{\lambda}\right) + \left(\frac{\lambda}{b}\right) + \frac{\pi^2 \sigma t b^2}{420D}\left(\frac{b}{\lambda}\right)\right] \quad (2)$$

where the first three terms belong to the elastic stiffness matrix and the fourth term is the geometrical stiffness. Dimensionally, Eq. (2) is different from that derived by Wittrick [Eq. (1)] because his expression is for a stress-couple (in lb-in./in.) while the finite element stiffness is for a node moment (in lb-in.). Noting that an iterative solution is required to determine the half-wavelength λ for the lowest stress σ in both the exact and finite element formulations of local instability analysis, the computational simplicity of the finite element coefficients, as typically represented by Eq. (2), is quite obvious.

Reference 1 gives typical computer times for the finite element computations, but unfortunately the corresponding times for the exact method are not available to make a meaningful comparison. It should be pointed out, however, that for many problems, the finite element solutions where each component flat is treated as a single element give sufficient accuracy for engineering purposes. Furthermore, since the author's method is based on the concept of geometrical stiffness, the method may be used directly in conjunction with any finite element computer programs for the over-all instability—a definite advantage for the design engineer who needs to investigate not only the local instability but also the over-all instability of the same configuration.

The restrictive assumption that the edge lines must remain straight during buckling (the classical assumption for local instability) used in Ref. 1 can be removed and the finite element method, developed by the author, can be extended to include the in-plane stiffnesses. In fact, work is presently underway to develop the necessary matrices. This extension would permit finite element studies of coupling between long wave and short wave (local) modes of instability which can presently be accomplished with Wittrick's exact method.

Finally, regarding the comment that some of the numerical results in Fig. 5 of Ref. 1 should be above the exact curve, it should be reaffirmed that the numerical results are correct. When the biaxial stress field is compressive (both σ_x and σ_y are positive) the convergence of finite element solutions is from above; however, when the σ_y stress is tensile (negative σ_y) and it is sufficiently high in relation to σ_x, the convergence is from below. The stabilizing influence of the tensile stress in the transverse direction on the strip apparently has a dominant effect on the convergence.

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References

[1] Przemieniecki, J. S., "Finite Element Structural Analysis of Local Instability," *AIAA Journal*, Vol. 11, No. 1, Jan. 1973, pp. 33–39.

[2] Wittrick, W. H., "A Unified Approach to the Initial Buckling of Stiffened Panels in Compression," *The Aeronautical Quarterly*, Vol. 19, Pt. 3, Aug. 1968, pp. 265–283.

Comment on "Hypersonic, Viscous Shock Layer with Chemical Nonequilibrium for Spherically Blunted Cones"

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IN a recent Note,[1] Kang and Dunn reported some of their numerical results for thin viscous shock-layer (TVSL) flows over spherically blunted cones. This Note was later supplemented

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